G ARITHMETIC SERIES

An **arithmetic series** is the sum of the terms of an arithmetic sequence.

G ARITHMETIC SERIES

An **arithmetic series** is the sum of the terms of an arithmetic sequence.

For example: $21, 23, 25, 27, ..., 49$ is a finite arithmetic sequence.

$21 + 23 + 25 + 27 + ... + 49$ is the corresponding arithmetic series.

G ARITHMETIC SERIES

An **arithmetic series** is the sum of the terms of an arithmetic sequence.

For example: $21, 23, 25, 27, ..., 49$ is a finite arithmetic sequence.

$21 + 23 + 25 + 27 + ... + 49$ is the corresponding arithmetic series.

SUM OF A FINITE ARITHMETIC SERIES

The sum of a finite arithmetic series with first term u_1, common difference d, and last term u_n, is

$$\boldsymbol{S_n = \frac{n}{2}(u_1 + u_n)} \qquad \textit{or} \qquad \boldsymbol{S_n = \frac{n}{2}(2u_1 + (n-1)d)}.$$

Rather than adding all the terms individually, we can use a formula to find the sum of a finite arithmetic series.

If the first term is u_1, the final term is u_n, and the common difference is d, the terms are $u_1,\ \ u_1 + d,\ \ u_1 + 2d,\ \ ....,\ \ (u_n - 2d),\ \ (u_n - d),\ \ u_n.$

$$\begin{aligned} \therefore \quad S_n &= u_1 + (u_1 + d) + (u_1 + 2d) + ... + (u_n - 2d) + (u_n - d) + u_n \\ \text{But} \quad S_n &= u_n + (u_n - d) + (u_n - 2d) + ... + (u_1 + 2d) + (u_1 + d) + u_1 \qquad \{\text{reversing them}\} \end{aligned}$$

Adding these two equations vertically, we get:

$$2S_n = \underbrace{(u_1 + u_n) + (u_1 + u_n) + (u_1 + u_n) + ... + (u_1 + u_n) + (u_1 + u_n) + (u_1 + u_n)}_{n \text{ of these}}$$

$$\therefore \quad 2S_n = n(u_1 + u_n)$$

$$\therefore \quad S_n = \frac{n}{2}(u_1 + u_n) \quad \text{where} \quad u_n = u_1 + (n - 1)d$$

Example 25 **Self Tutor**

Find the sum of $4+7+10+13+....$ to 50 terms.

Example 25 **Self Tutor**

Find the sum of $4+7+10+13+\ldots.$ to 50 terms.

The series is arithmetic with $u_1 = 4$, $d = 3$, and $n = 50$.

Now $S_n = \frac{n}{2}(2u_1 + (n-1)d)$

$$\begin{aligned} \therefore\ S_{50} &= \tfrac{50}{2}(2 \times 4 + 49 \times 3) \\ &= 3875 \end{aligned}$$

Example 26

Self Tutor

Find the sum of $-6 + 1 + 8 + 15 + ... + 141$.

Example 26 **Self Tutor**

Find the sum of $-6 + 1 + 8 + 15 + ... + 141$.

The series is arithmetic with $u_1 = -6$, $d = 7$, and $u_n = 141$.

First we need to find n.

$$\begin{aligned} \text{Now} \quad u_n &= 141 \\ \therefore \quad u_1 + (n-1)d &= 141 \\ \therefore \quad -6 + 7(n-1) &= 141 \\ \therefore \quad 7(n-1) &= 147 \\ \therefore \quad n - 1 &= 21 \\ \therefore \quad n &= 22 \end{aligned}$$

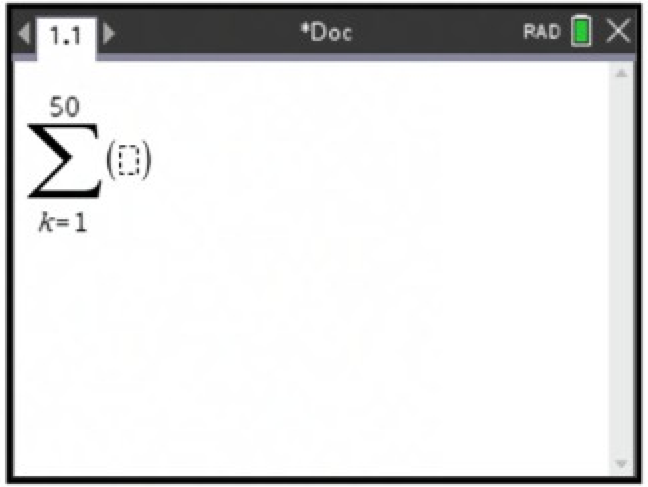
Using $S_n = \frac{n}{2}(u_1 + u_n)$,

$$\begin{aligned} S_{22} &= \tfrac{22}{2}(-6 + 141) \\ &= 11 \times 135 \\ &= 1485 \end{aligned}$$

We can write the series $4 + 7 + 10 + ...$ to 50 terms as $\sum_{k=1}^{50} (3k + 1)$.

To evaluate $\sum_{k=1}^{50} (3k+1)$, press **menu**, select **4 Calculus**, then select **3 Sum**.

First enter the limits of the sum by pressing **K** ▶ 1 ▲ ▲ 50.

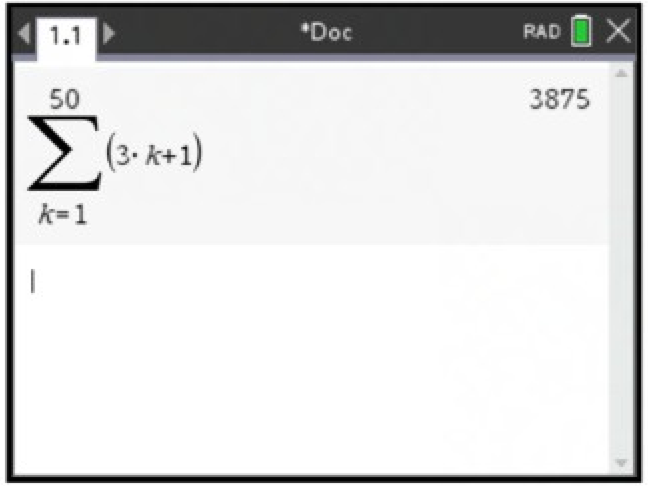


$$\sum_{k=1}^{50} (\square)$$

Then enter the formula in the brackets by pressing ▶ 3 **K** **+** 1.

Press **enter** to evaluate the sum.

So, $\sum_{k=1}^{50} (3k + 1) = 3875$.



$$\sum_{k=1}^{50} (3 \cdot k+1) \qquad 3875$$

1 Find the sum of:

- **a** $7 + 9 + 11 + 13 + ...$ to 10 terms
- **b** $3 + 7 + 11 + 15 + ...$ to 20 terms
- **c** $\frac{1}{2} + 3 + 5\frac{1}{2} + 8 + ...$ to 50 terms
- **d** $100 + 93 + 86 + 79 + ...$ to 40 terms
- **e** $(-31) + (-28) + (-25) + (-22) + ...$ to 15 terms
- **f** $50 + 48\frac{1}{2} + 47 + 45\frac{1}{2} + ...$ to 80 terms.

2 Find the sum of:

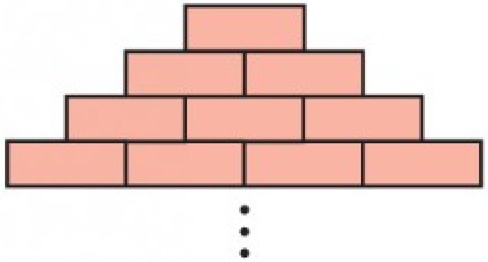
- **a** $5 + 8 + 11 + 14 + ... + 101$
- **b** $37 + 33 + 29 + 25 + ... + 9$
- **c** $50 + 49\frac{1}{2} + 49 + 48\frac{1}{2} + ... + (-20)$
- **d** $8 + 10\frac{1}{2} + 13 + 15\frac{1}{2} + ... + 83$

3 Evaluate these arithmetic series:

- **a** $\sum_{k=1}^{10} (2k + 5)$
- **b** $\sum_{k=1}^{15} (k - 50)$
- **c** $\sum_{k=1}^{20} \left(\frac{k+3}{2}\right)$

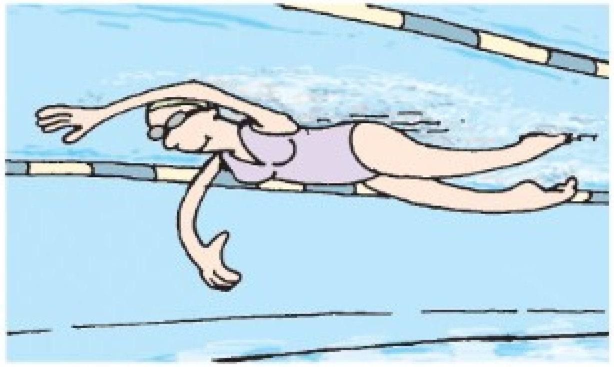
Check your answers using technology.

4 An arithmetic series has eleven terms. The first term is 6 and the last term is -27. Find the sum of the series.

5 

A bricklayer builds a triangular wall with layers of bricks as shown. If the bricklayer uses 171 bricks, how many layers did he build?

6 Vicki has 30 days to train for a swimming competition. She swims 20 laps on the first day, then each day after that she swims two more laps than the previous day. So, she swims 22 laps on the second day, 24 laps on the third day, and so on.



a How many laps does Vicki swim on:

  i the tenth day **ii** the final day?

b How many laps does Vicki swim in total?

7 A woman deposits \$100 into her son's savings account on his first birthday. She deposits \$125 on his second birthday, \$150 on his third birthday, and so on.

a Calculate the amount of money she will deposit into her son's account on his 15th birthday.

b Find the total amount she will have deposited over the 15 years.

8 A football stadium has 25 sections of seating. Each section has 44 rows of seats, with 22 seats in the first row, 23 in the second row, 24 in the third row, and so on. How many seats are there in:

- **a** row 44 of one section
- **b** each section
- **c** the whole stadium?

9 Find the sum of:

- **a** the first 50 multiples of 11
- **b** the multiples of 7 between 0 and 1000
- **c** the integers from 1 to 100 which are not divisible by 3
- **d** the three-digit numbers which start or end with a "4".

10 $k-1$, $2k+3$, and $27-k$ are the first three terms of an arithmetic sequence.

- **a** Find k.
- **b** Find the sum of the first 25 terms of the sequence.

11 The sixth term of an arithmetic sequence is 21, and the sum of the first seventeen terms is 0. Find the first two terms of the sequence.

12 An arithmetic series has $S_3 = 9$ and $S_6 = 90$. Find S_{10}.

Example 27 **Self Tutor**

An arithmetic sequence has first term 8 and common difference 2. The sum of the terms of the sequence is 170. Find the number of terms in the sequence.

Example 27 **Self Tutor**

An arithmetic sequence has first term 8 and common difference 2. The sum of the terms of the sequence is 170. Find the number of terms in the sequence.

Now $S_n = 170$, so

$$\begin{aligned} \frac{n}{2}(2u_1 + (n-1)d) &= 170 \\ \therefore\ \frac{n}{2}(16 + 2(n-1)) &= 170 \\ \therefore\ 8n + n(n-1) &= 170 \\ \therefore\ n^2 + 7n - 170 &= 0 \\ \therefore\ (n+17)(n-10) &= 0 \\ \therefore\ n &= 10 \quad \{\text{as } n > 0\} \end{aligned}$$

$\therefore$ there are 10 terms in the sequence.

13 An arithmetic sequence has first term 4 and common difference 6. The sum of the terms of the sequence is 200. Find the number of terms in the sequence.

14 An arithmetic sequence has $u_1 = 7$ and $S_2 = 17$.

- **a** Find the common difference of the sequence.
- **b** Find n such that $S_n = 242$.

15 Consider the arithmetic sequence 13, 21, 29, 37, ... How many terms are needed for the sum of the sequence terms to exceed 1000?

16 Use the arithmetic sequence formula to prove that the sum of the first n integers is $\frac{n(n+1)}{2}$.

17
- **a** Write a formula for the nth odd integer.
- **b** Hence prove that the sum of the first n odd integers is n^2.

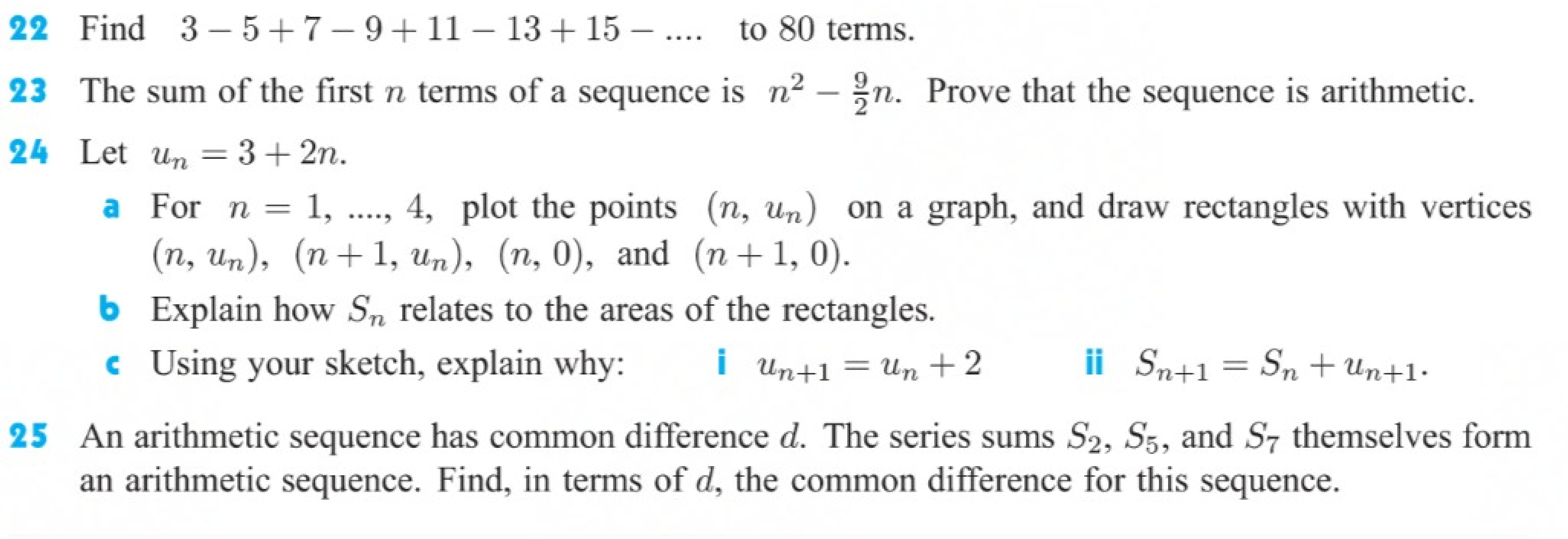
18 Three consecutive terms of an arithmetic sequence have a sum of 12 and a product of -80. Find the terms.

19 The sum of the first 15 terms of an arithmetic sequence is 480. Find the 8th term of the sequence.

20 Five consecutive terms of an arithmetic sequence have a sum of 40. The product of the first, middle, and last terms is 224. Find the terms of the sequence.

21 The sum of the first n terms of an arithmetic sequence is $\frac{n(3n+11)}{2}$.

- **a** Find its first two terms.
- **b** Find the twentieth term of the sequence.



22 Find $3 - 5 + 7 - 9 + 11 - 13 + 15 - ....$ to 80 terms.

23 The sum of the first n terms of a sequence is $n^2 - \frac{9}{2}n$. Prove that the sequence is arithmetic.

24 Let $u_n = 3 + 2n$.

- **a** For $n = 1, ..., 4$, plot the points (n, u_n) on a graph, and draw rectangles with vertices (n, u_n), $(n+1, u_n)$, $(n, 0)$, and $(n+1, 0)$.
- **b** Explain how S_n relates to the areas of the rectangles.
- **c** Using your sketch, explain why: **i** $u_{n+1} = u_n + 2$ **ii** $S_{n+1} = S_n + u_{n+1}$.

25 An arithmetic sequence has common difference d. The series sums S_2, S_5, and S_7 themselves form an arithmetic sequence. Find, in terms of d, the common difference for this sequence.

1 **a** 160 **b** 820 **c** $3087\frac{1}{2}$ **d** -1460
e -150 **f** -740

2 **a** 1749 **b** 184 **c** 2115 **d** $1410\frac{1}{2}$

3 **a** 160 **b** -630 **c** 135

4 $-115\frac{1}{2}$ **5** 18 layers

6 **a** **i** 38 laps **ii** 78 laps **b** 1470 laps

7 **a** \$450 **b** \$4125

8 **a** 65 seats **b** 1914 seats **c** 47 850 seats

9 **a** 14 025 **b** 71 071 **c** 3367 **d** 89 870

10 **a** $k = 5$ **b** $S_{25} = 2800$

11 $u_1 = 56,\ u_2 = 49$ **12** $S_{10} = 310$ **13** 8 terms

14 **a** $d = 3$ **b** $n = 11$ **15** 15 terms

16 **Hint:** Write out the terms of the sequence in order, then write out the terms in reverse order directly underneath. Add the terms vertically.

17 **a** $u_n = 2n - 1$

b $S_n = \frac{n}{2}(u_1 + u_n) = \frac{n}{2}(1 + 2n - 1) = n^2$

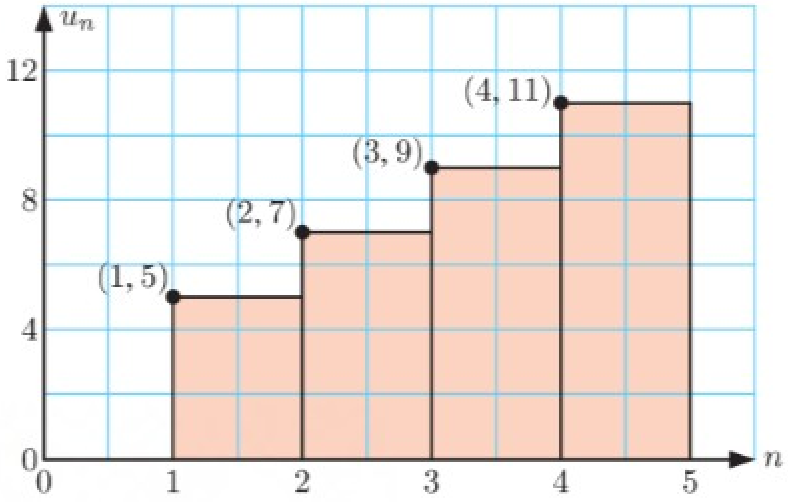
18 10, 4, -2 or -2, 4, 10 **19** $u_8 = 32$

20 2, 5, 8, 11, 14 or 14, 11, 8, 5, 2

21 **a** $u_1 = 7,\ u_2 = 10$ **b** $u_{20} = 64$

22 $S_{80} = -80$ **23** $u_1 = -\frac{7}{2},\ d = 2$

24 **a**



b S_n is the sum of the areas of the first n rectangles.

c **i** The height of each rectangle increases by 2 units from the previous rectangle, so $u_{n+1} = u_n + 2$.

ii The area of the $(n+1)$ th rectangle is u_{n+1}.
S_{n+1} is the sum of the areas of the first n rectangles and the $(n+1)$ th rectangle, $S_{n+1} = S_n + u_{n+1}$.

25 $15d$